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Generalized transvectants and Siegel modular forms

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Abstract

We introduce a differential operator invariant under the special linear group $\mathrm{SL}(2n,\mathbb{C})$, and, as a consequence, the symplectic group $\mathrm{Sp}(2n,\mathbb{C})$. Connections with generalized Rankin–Cohen brackets for Siegel modular forms of genus n are sketched.

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1. Introduction

An important differential operator in the context of modular forms is the Rankin–Cohen bracket [6,14]. An analogy with the transvectant of classical invariant theory was noted in [15]. An explanation of this analogy is given in [5,10–12]. The fundamental idea is to compare the transformation law for modular forms under a fixed group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ with the action of Γ on homogeneous polynomials written in projective coordinates. We adopt a geometric viewpoint. It transpires that the Rankin–Cohen bracket is exactly a transvectant operator acting on homogeneous functions of negative degree in one projective coordinate.

Recently generalizations of this bracket to Siegel modular forms have been introduced in [3,4,7,8]. The present work is an attempt to enlarge the philosophy of [5,11,12] to that wider context. In Section 2.2, we will compare the transformation law of Siegel modular forms of genus n under

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a fixed group $\Gamma \subset \mathrm{Sp}(2n,\mathbb{R})$ with the diagonal action of Γ on homogeneous functions of two matrix variables of $\mathfrak{gl}(n,\mathbb{C})$. In Section 3, we generalize the classical transvectant operator to complex functions of matrix domain (Theorem 1), providing thereby a Rankin–Cohen bracket for Siegel modular forms (Corollary 1). Explicit examples of vectorial/symplectic transvectants appear in Section 4, while Section 5 develops an infinitesimal calculus for use in more general contexts.

2. Definitions and notations

2.1. Projective representative of homogeneous functions

Let $\mathfrak{gl}(n,\mathbb{C})$ denote the $\mathbb{C}$-algebra of complex n by n matrices. For any $k \in \mathbb{Z}$, we will say that a meromorphic function

$$F : \mathfrak{gl}(n,\mathbb{C}) \times \mathfrak{gl}(n,\mathbb{C}) \longrightarrow \mathbb{C}$$

is *homogeneous* of degree k if and only if it transforms as

$$F(X\Lambda, Y\Lambda) = (\det \Lambda)^k F(X,Y), \quad X, Y \in \mathfrak{gl}(n,\mathbb{C}) \quad \text{for all } \Lambda \in \mathrm{GL}(n,\mathbb{C}). \tag{1}$$

For instance, $\det(X+Y)$ is homogeneous of degree 1.

Let $\mathcal{P}$ denote the field of complex meromorphic functions with domain $\mathfrak{gl}(n,\mathbb{C})$. Consider the graded ring

$$\mathcal{H}_\bullet := \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k,$$

where $\mathcal{H}_k$ denotes the vector space of homogeneous complex functions of degree k. The *projective representative* $f \in \mathcal{P}$ of a function $F \in \mathcal{H}_k$ is defined by the formula

$$f(Z) := F(Z, \mathrm{I}_n), \quad Z \in \mathfrak{gl}(n,\mathbb{C}), \tag{2}$$

where I_n denotes the identity matrix, and satisfies, by the homogeneity of F, the relation

$$F(X,Y) = (\det Y)^k f\big(XY^{-1}\big), \quad X, Y \in \mathfrak{gl}(n,\mathbb{C}). \tag{3}$$

Formulae (2) and (3) establish a bijective correspondence

$$\pi_k : \mathcal{H}_k \longrightarrow \mathcal{P} \quad \text{where } \pi_k(F) = f. \tag{4}$$

The right action of the group $\mathrm{GL}(2n,\mathbb{C})$ on the space $\mathcal{H}_k$ is defined by pull-back on letting:

$$(F \circ \gamma)(X,Y) := F(AX + BY, CX + DY) \tag{5}$$

where $A, B, C, D \in \mathfrak{gl}(n,\mathbb{C})$ and $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}(2n,\mathbb{C})$.

To each $k \in \mathbb{Z}$, we attach the (anti)representation

$$\rho_k : \mathrm{GL}(2n,\mathbb{C}) \longrightarrow \mathrm{End}\,\mathcal{P} \tag{6}$$

by letting $\rho_k(\gamma)f := \pi_k(F \circ \gamma)$. A direct computation whose result will be used later yields:

$$\rho_k(\gamma)f(Z) = \det(CZ + D)^k f\big((AZ + B)(CZ + D)^{-1}\big). \tag{7}$$

Since $\mathrm{GL}(2n, \mathbb{C})$ acts on the right on $\mathcal{P}$, we obtain

$$\rho_k(\gamma_1\gamma_2) = \rho_k(\gamma_2)\rho_k(\gamma_1) \quad \text{for all } k \in \mathbb{Z} \text{ and } \gamma_1, \gamma_2 \in \mathrm{GL}(2n, \mathbb{C}). \tag{8}$$

A function $f \in \mathcal{P}$ is said to be γ-invariant by the representation ρ_k when

$$f\big((AZ + B)(CZ + D)^{-1}\big) = \det(CZ + D)^{-k} f(Z). \tag{9}$$

2.2. Siegel modular forms

The *symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$ is usually defined in geometry as the set of matrices $M \in \mathrm{GL}(2n, \mathbb{R})$ that preserve the canonical symplectic form $\omega = \sum_{i=1}^n dx^i \wedge dx^{n+i}$. Thus, $M \in \mathrm{Sp}(2n, \mathbb{R})$ if and only if

$$M^t \mathrm{J}_n M = \mathrm{J}_n, \tag{10}$$

where M^t denotes its transpose, and J_n is the $2n \times 2n$ matrix given by

$$\mathrm{J}_n = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

In number theory it is customary to write $M \in \mathrm{Sp}(2n, \mathbb{R})$ in block form as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the 4 matrices A, B, C, D satisfy three relations

$$A^t C = C^t A, \qquad B^t D = D^t B, \qquad A^t D - C^t B = \mathrm{I}_n \tag{11}$$

or, equivalently, that the inverse of M is given by a block version of the formula for the inverse of a matrix in SL(2):

$$M^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}.$$

Conversely it is straightforward to check that every such M belongs to $\mathrm{Sp}(2n, \mathbb{R})$. Note that $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$, since every matrix that preserves the canonical symplectic form also clearly preserves the volume form

$$dx^1 \wedge \cdots \wedge dx^{2n} = \frac{1}{n!}\omega \wedge \cdots \wedge \omega.$$

The space of all n by n complex symmetric matrices with positive definite imaginary part is usually called the *Siegel upper half-space* $\mathbb{H}_n$, the analogue of the Poincaré upper half-plane for $n=1$. The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ acts on the Siegel upper half-space by the rule

$$Z \mapsto (AZ+B)(CZ+D)^{-1}.$$

A proof of the basic fact that $\mathrm{Sp}(2n,\mathbb{R})$ preserves $\mathbb{H}_n$ can be found, for instance, in [1].

Let $\Gamma \subset \mathrm{Sp}(2n,\mathbb{R})$ be a discrete subgroup of finite co-volume, meaning that the volume of $\mathrm{Sp}(2n,\mathbb{R})/\Gamma$ is finite.

Definition 1. A *Siegel modular form* of weight $k \geqslant 0$ and genus n on Γ is any holomorphic function $f(Z)$ of $Z \in \mathbb{H}_n$ which transforms under all $\gamma \in \Gamma$ as

$$f\big((AZ+B)(CZ+D)^{-1}\big) = \det(CZ+D)^k f(Z) \quad \text{where } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{12}$$

We denote the space of all Siegel modular forms on Γ by $\mathcal{M}_\bullet\Gamma$. This space is equipped with the structure of a graded ring, the grading being provided by the weight k:

$$\mathcal{M}_\bullet\Gamma = \mathcal{M}_0\Gamma \oplus \mathcal{M}_1\Gamma \oplus \mathcal{M}_2\Gamma \oplus \cdots.$$

In the case $n=1$, condition (11) demonstrates that $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$, and $\mathcal{M}_\bullet\Gamma$ coincides with the space of classical *modular forms*.

Comparing the invariance conditions (9) and (12) suggests the fundamental observation :

A Siegel modular form of weight k transforms as the projective representative of a homogeneous function of degree $-k$.

3. Generalized transvectants

3.1. Computation in the affine case

Let Γ denote a fixed subgroup of $\mathrm{SL}(2n,\mathbb{C})$. Let $\mathcal{H}_k^\Gamma$ denote the subspace of Γ-invariant elements of $\mathcal{H}_k$. By (5), these functions $F \in \mathcal{H}_k^\Gamma$ satisfy

$$F(AX+BY, CX+DY) = F(X,Y), \quad \text{for all} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma. \tag{13}$$

The product of two Γ-invariant functions of degree k and l is also a Γ-invariant function of degree $k+l$. Hence $\mathcal{H}_\bullet^\Gamma := \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k^\Gamma$ is a graded ring. The purpose of this section is to construct explicit $\mathrm{SL}(2n,\mathbb{C})$-covariant operators from $\mathcal{H}_k^\Gamma \otimes \mathcal{H}_l^\Gamma \to \mathcal{H}_m^\Gamma$, which will then restrict to Rankin–Cohen brackets on the spaces of Siegel modular forms.

We define the *omega operator* by the formula

$$\Omega := \det \begin{pmatrix} \partial_{X^1} & \partial_{X^2} \\ \partial_{Y^1} & \partial_{Y^2} \end{pmatrix} \quad \text{where } X^1, Y^1, X^2, Y^2 \in \mathfrak{gl}(n,\mathbb{C}), \tag{14}$$

where ∂_X means the matrix of operators $(\partial_{X_{i,j}})$, where i (respectively j) is a row (respectively column) index. This operator acts on functions from $\mathfrak{gl}(2n, \mathbb{C})$ to $\mathbb{C}$.

Example. In the case of genus $n = 2$, we have matrix variables

$$X^i = \begin{pmatrix} x_{11}^{(i)} & x_{12}^{(i)} \\ x_{21}^{(i)} & x_{22}^{(i)} \end{pmatrix}, \qquad Y^i = \begin{pmatrix} y_{11}^{(i)} & y_{12}^{(i)} \\ y_{21}^{(i)} & y_{22}^{(i)} \end{pmatrix},$$

for $i = 1, 2$, and so

$$\Omega = \det \begin{pmatrix} \partial_{x_{11}}^{(1)} & \partial_{x_{12}}^{(1)} & \partial_{x_{11}}^{(2)} & \partial_{x_{12}}^{(2)} \\ \partial_{x_{21}}^{(1)} & \partial_{x_{22}}^{(1)} & \partial_{x_{21}}^{(2)} & \partial_{x_{22}}^{(2)} \\ \partial_{y_{11}}^{(1)} & \partial_{y_{12}}^{(1)} & \partial_{y_{11}}^{(2)} & \partial_{y_{12}}^{(2)} \\ \partial_{y_{21}}^{(1)} & \partial_{y_{22}}^{(1)} & \partial_{y_{21}}^{(2)} & \partial_{y_{22}}^{(2)} \end{pmatrix}. \tag{15}$$

Definition 2. Let $\Omega^r := \Omega \circ \Omega \circ \cdots \circ \Omega$ denote the rth power of the operator Ω. The *transvectant* $[F, G]_r$ of order r of two homogeneous functions F, G is then defined for all $r \in \mathbb{N}$ as

$$[F, G]_r(X, Y) = \Omega^r\big(F\big(X^1, Y^1\big)G\big(X^2, Y^2\big)\big)\big|_{X^1=X^2=X, Y^1=Y^2=Y}. \tag{16}$$

We can now state our main result.

Theorem 1. *Let Γ denote a subgroup of* $\mathrm{SL}(2n, \mathbb{C})$. *Then, for all functions $F \in \mathcal{H}_k^\Gamma$ and $G \in \mathcal{H}_l^\Gamma$, we have $[F, G]_r \in \mathcal{H}_{k+l-2r}^\Gamma$.*

Corollary 1. *For all $r \in \mathbb{N}$, the rth transvectant* (24) *restricts to a Rankin–Cohen bracket on the space of Siegel modular forms.*

To begin the proof, we first establish the invariance of the omega operator.

Proposition 1. *The omega operator is invariant under* $\mathrm{SL}(2n, \mathbb{C})$.

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

a typical element of $\mathrm{SL}(2n, \mathbb{C})$. Assume

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} := M \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Differentiating we obtain

$$\begin{pmatrix} d\bar{X} \\ d\bar{Y} \end{pmatrix} = M \begin{pmatrix} dX \\ dY \end{pmatrix}.$$

By duality,

$$\begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix} = M^t \begin{pmatrix} \partial_{\bar{X}} \\ \partial_{\bar{Y}} \end{pmatrix}$$

which yields

$$\begin{pmatrix} \partial_{X^1} & \partial_{X^2} \\ \partial_{Y^1} & \partial_{X^2} \end{pmatrix} = M^t \begin{pmatrix} \partial_{\bar{X}^1} & \partial_{\bar{X}^2} \\ \partial_{\bar{Y}^1} & \partial_{\bar{Y}^2} \end{pmatrix}$$

and hence $\Omega = \det(M)\ \bar{\Omega}$. Since we are assuming $\det(M) = 1$, we conclude that $\bar{\Omega} = \Omega$. $\square$

In what follows, all the tensor products are computed over $\mathbb{C}$. The $\mathbb{C}$-algebra $\mathcal{H}_\bullet \otimes \mathcal{H}_\bullet$ is equipped with the product

$$(F \otimes G)(F' \otimes G') = (FF') \otimes (GG').$$

This algebra is graded by the degree *deg* by letting $\deg(F \otimes G) = \deg F + \deg G$. The operator Ω is regarded here as the $\mathbb{C}$-linear map

$$\Omega : \mathcal{H}_\bullet \otimes \mathcal{H}_\bullet \longrightarrow \mathcal{H}_\bullet \otimes \mathcal{H}_\bullet.$$

(The fact that Ω preserves $\mathcal{H}_\bullet \otimes \mathcal{H}_\bullet$ will follow from Corollary 2 below.)

The Laplace formula is a well-known expansion of the determinant of a block matrix. The operator Ω acts on the $n \times n$ blocks of a $2n \times 2n$ matrix according to

$$\Omega = \sum_\sigma (-1)^{\operatorname{sgn}(\sigma)} D_{i_1,i_2,\dots,i_n} \otimes D_{j_1,j_2,\dots,j_n} \tag{17}$$

where the summation range is over all shuffles of type (n, n), that is, all permutations $\sigma \in \mathfrak{S}_{2n}$ of the form

$$\sigma = (i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n) \quad \text{with } i_1 < i_2 < \cdots < i_n, \quad j_1 < j_2 < \cdots < j_n.$$

The determinant $D_{i_1,i_2,\dots,i_n}$ is obtained from the matrices of operators $\binom{\partial_X}{\partial_Y}$ by selecting the n distinct rows $i_1, i_2, \dots, i_n$.

Example. For instance in the case of (15) we obtain

$$\Omega = (D_{12} \otimes D_{34} + D_{34} \otimes D_{12}) - (D_{13} \otimes D_{24} + D_{24} \otimes D_{13}) + (D_{14} \otimes D_{23} + D_{23} \otimes D_{14}) \tag{18}$$

with $D_{12} = \partial_{x_{11}} \partial_{x_{22}} - \partial_{x_{12}} \partial_{x_{21}}$, $D_{13} = \partial_{x_{11}} \partial_{y_{12}} - \partial_{x_{12}} \partial_{y_{11}}$, and so on.

Lemma 1. *Keeping the notation of* (17), $D_{i_1,i_2,\dots,i_n}$ *is a homogeneous differential operator of degree* -1, *viz.*

$$D_{i_1,i_2,\dots,i_n} : \mathcal{H}_k \longrightarrow \mathcal{H}_{k-1}$$

for all $k \in \mathbb{Z}$.

Proof. We first give a proof in the case $n = 1$. We need to verify that the functions F_x (respectively, F_y) are homogeneous of degree $k - 1$ whenever F is homogeneous of degree k. We claim that

$$F_x(\lambda x, \lambda y) = \lambda^{k-1} F_x(x, y). \tag{19}$$

Indeed, by the relation $F(\lambda x, \lambda y) = \lambda^k F(x, y)$ and the *chain rule*

$$\partial_x\big(F(\lambda x, \lambda y)\big) = \lambda F_x(\lambda x, \lambda y), \tag{20}$$

the result follows. The relation (20) can be reworded as

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \lambda \quad \Longleftrightarrow \quad \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \partial_{\bar{x}} \\ \partial_{\bar{y}} \end{pmatrix} \lambda. \tag{21}$$

We generalize this relation (21) to the situation of genus n by replacing the complex variables x and y by matrices $X, Y \in \mathfrak{gl}(n, \mathbb{C})$ as well as $\lambda \in \mathbb{C}$ by a matrix $\Lambda \in \mathrm{GL}(n, \mathbb{C})$. We obtain

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \Lambda \quad \Longleftrightarrow \quad \begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix} = \begin{pmatrix} \partial_{\bar{X}} \\ \partial_{\bar{Y}} \end{pmatrix} \Lambda^t. \tag{22}$$

For a subdeterminant $D := D_{i_1, i_2, \ldots, i_n}$ of $\left(\begin{smallmatrix} \partial_X \\ \partial_Y \end{smallmatrix}\right)$, we derive

$$D = \det(\Lambda) \bar{D}. \tag{23}$$

Thus, for $F \in \mathcal{H}_k$, we generalize(19) in the form

$$DF(X\Lambda, Y\Lambda) = (\det \Lambda)^{k-1} DF(X, Y). \qquad \square$$

Corollary 2. *For all $k, l \in \mathbb{Z}$, the operator Ω maps $\mathcal{H}_k \otimes \mathcal{H}_l$ onto $\mathcal{H}_{k-1} \otimes \mathcal{H}_{l-1}$.*

Proof. This follows from the Laplace formula (17) for the determinant Ω and Lemma 1. $\square$

The product $\mu(F \otimes G) = FG$ of two homogeneous functions F, G of degree k, l is regarded as a linear map

$$\mu : \mathcal{H}_k \otimes \mathcal{H}_l \longrightarrow \mathcal{H}_{k+l}.$$

Hence, the order r transvectant of two homogeneous functions F and G can be written as

$$[F, G]_r = \mu \circ \Omega^r (F \otimes G). \tag{24}$$

Proof of Theorem 1. As Ω is Γ-invariant, then, for all $r \in \mathbb{N}$, Ω^r is Γ-invariant and, by Proposition 1 maps $\mathcal{H}_k \otimes \mathcal{H}_l$ into $\mathcal{H}_{k-r} \otimes \mathcal{H}_{l-r}$. We conclude by noticing that $\mu(\mathcal{H}_{k-r} \otimes \mathcal{H}_{l-r})^\Gamma \subset \mathcal{H}^\Gamma_{k+l-2r}$. $\square$

3.2. Transvectants in the projective representation

Let $\mathcal{P}$ denote the set of smooth functions from $\mathfrak{gl}(n,\mathbb{C})$ into $\mathbb{C}$. The homogeneous component $\mathcal{P}_k^\Gamma$ consists—see (7)—of functions that are invariant under the representation ρ_k:

$$\mathcal{P}_k^\Gamma := \{ f \in \mathcal{P} \mid \rho_k(\gamma) f = f \text{ for all } \gamma \in \Gamma \}. \tag{25}$$

The direct sum $\mathcal{P}_\bullet^\Gamma := \bigoplus_{k \in Z} \mathcal{P}_k^\Gamma$ is a graded ring.

The differential operator $\omega_{k,l} : \mathcal{P} \otimes \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{P}$ is defined as the conjugate of the operator Ω:

$$\begin{tikzcd} \mathcal{H}_k \otimes \mathcal{H}_l \arrow[r, "\Omega"] \arrow[d, "\pi_k \otimes \pi_l"'] & \mathcal{H}_{k-1} \otimes \mathcal{H}_{l-1} \arrow[d, "\pi_{k-1} \otimes \pi_{l-1}"] \\ \mathcal{P} \otimes \mathcal{P} \arrow[r, "\omega_{k,l}"] & \mathcal{P} \otimes \mathcal{P} \end{tikzcd} . \tag{26}$$

The formula (24) becomes for all $f \in \mathcal{P}_k^\Gamma$ and $g \in \mathcal{P}_l^\Gamma$

$$[f,g]_r = \mu \circ \underbrace{\omega_{k-r+1,\,l-r+1} \circ \cdots \circ \omega_{k-1,\,l-1} \circ \omega_{k,\,l}}_{r}(f \otimes g). \tag{27}$$

Proposition 2. *Let Γ denote a subgroup of* $\mathrm{SL}(2n,\mathbb{C})$. *Then for all functions $f \in \mathcal{P}_k^\Gamma$ and $g \in \mathcal{P}_l^\Gamma$, we have that $[f,g]_r \in \mathcal{P}_{k+l-2r}^\Gamma$.*

4. Explicit computations of transvectants

4.1. Transvectants for genus $n = 1$

When $n = 1$, the formula (14) reduces to the classical Cayley omega process $\Omega = \partial_{x^1}\partial_{y^2} - \partial_{y^1}\partial_{x^2}$ [11]. Using tensorial formalism, we have

$$\Omega = \partial_x \otimes \partial_y - \partial_y \otimes \partial_x. \tag{28}$$

For instance, the transvectant of order $r = 1$ coincides with the *Poisson bracket*:

$$[F,G]_1 = F_x G_y - F_y G_x.$$

Since the operators $\partial_x \otimes \partial_y$ and $\partial_y \otimes \partial_x$ commute, we can apply Newton's binomial formula to obtain for all $r \in \mathbb{N}$

$$(\partial_x \otimes \partial_y - \partial_y \otimes \partial_x)^r = \sum_{i+j=r} (-1)^j \binom{r}{i} \partial_x^i \partial_y^j \otimes \partial_x^j \partial_y^i,$$

$$[F,G]_r = \sum_{i+j=r} (-1)^j \binom{r}{i} \frac{\partial^r F}{\partial x^i \partial y^j} \frac{\partial^r G}{\partial x^j \partial y^i}.$$

We proceed to compute the order r transvectant on $\mathcal{P}^{\Gamma}$, that is for functions of the variable $z = x/y$. This yields for $F(x,y) = y^k\, f(x/y)$ the formulas

$$\begin{cases} F_x(x,y) = y^{k-1} f_z(z), \\ F_y(x,y) = k y^{k-1} f(z) - z y^{k-1} f_z(z). \end{cases} \tag{29}$$

If $F \in \mathcal{H}_k$ then $F_x, F_y \in \mathcal{H}_{k-1}$. The derivations ∂_x and ∂_y commute. By (29) and (3), they correspond to the differential operators

$$\begin{cases} \partial_z = \pi_{k-1} \circ \partial_x \circ \pi_k^{-1}, \\ k - z\partial_z = \pi_{k-1} \circ \partial_y \circ \pi_k^{-1}. \end{cases} \tag{30}$$

These differential operators do not commute:

$$\partial_z(k - z\partial_z) = (k - 1 - z\partial_z)\partial_z \quad \text{and so} \quad [\partial_z, k - z\partial_z] = -\partial_z. \tag{31}$$

Therefore, from (28) and (30)

$$\omega_{k,l} = \partial_z \otimes (l - z\partial_z) - (k - z\partial_z) \otimes \partial_z = \partial_z \otimes l - k \otimes \partial_z + z\partial_z \otimes \partial_z - \partial_z \otimes z\partial_z.$$

The order 1 transvectant evaluates as

$$[f, g]_1 = l f_z g - k f g_z$$

since the term $z\partial_z \otimes \partial_z - \partial_z \otimes z\partial_z$ is annihilated by the multiplication operator μ:

$$\mu \circ (z\partial_z \otimes \partial_z - \partial_z \otimes z\partial_z)(f \otimes g) = \mu(z f_z \otimes g_z - f_z \otimes z g_z) = z f_z g_z - f_z z g_z = 0.$$

We shall require the falling factorial notation

$$n^{\underline{k}} := n(n-1)\cdots(n-k+1).$$

Combining Newton binomial formula with the said commutation rules (31) yields, for the order r transvectant, the expression

$$[f, g]_r = \sum_{i+j=r} (-1)^j \binom{r}{j} (k-i)^{\underline{j}} (l-j)^{\underline{i}} f^{(i)} g^{(j)} \tag{32}$$

which coincides with equation (2.12) in [12], namely

$$[f, g]_r = r! \sum_{i+j=r} (-1)^j \binom{k-i}{j} \binom{l-j}{i} f^{(i)} g^{(j)} \tag{33}$$

on identifying

$$\binom{k-i}{j} = \frac{(k-i)^{\underline{j}}}{j!}, \qquad \binom{l-j}{i} = \frac{(l-j)^{\underline{i}}}{i!}, \qquad \binom{r}{j} = \frac{r!}{i!\,j!}.$$

4.2. Transvectants for genus $n = 2$

4.2.1. Coproduct of a differential operator

The *coproduct* Δ of a constant coefficient differential operator D is defined as

$$\begin{cases} \Delta(D) = D \otimes 1 + 1 \otimes D \quad \text{for } D \text{ a derivation}, \\ \Delta(\lambda_1 D_1 + \lambda_2 D_2) = \lambda_1 \Delta(D_1) + \lambda_2 \Delta(D_2), \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C}, \\ \Delta(D_1 D_2) = \Delta(D_1)\Delta(D_2). \end{cases} \tag{34}$$

By the Leibniz rule, we have

$$\mu \circ \Delta(D) = D \circ \mu.$$

Thus, the coproduct can be decomposed into its homogeneous constituents of order $i, j \in \mathbb{N}$:

$$\Delta(D) = \bigoplus_{i,j\in\mathbb{N}} \Delta_{i,j}(D). \tag{35}$$

For example, given $D = \partial_{z_{11}}\partial_{z_{22}} - \partial_{z_{12}}\partial_{z_{21}}$, we obtain

$$\begin{cases} \Delta_{2,0} D = D \otimes 1, \\ \Delta_{1,1} D = \partial_{z_{11}} \otimes \partial_{z_{22}} + \partial_{z_{22}} \otimes \partial_{z_{11}} - \partial_{z_{12}} \otimes \partial_{z_{21}} - \partial_{z_{21}} \otimes \partial_{z_{12}}, \\ \Delta_{0,2} D = 1 \otimes D. \end{cases} \tag{36}$$

4.2.2. The computation of $\omega_{k,l}$

Let

$$d_{i,j}(k) = \pi_{k-1} \circ D_{i,j} \circ \pi_k^{-1}.$$

Then the determinantal Laplace formula (18) becomes

$$\begin{aligned} \omega_{k,l} = {} & \big(d_{12}(k) \otimes d_{34}(l) + d_{34}(k) \otimes d_{12}(l)\big) - \big(d_{13}(k) \otimes d_{24}(l) + d_{24}(k) \otimes d_{13}(l)\big) \\ & + \big(d_{14}(k) \otimes d_{23}(l) + d_{23}(k) \otimes d_{14}(l)\big). \end{aligned} \tag{37}$$

For a degree k function, computer algebra calculations yield

$$\begin{cases} d_{12}(k) = D \text{ where } D := \partial_{z_{11}}\partial_{z_{22}} - \partial_{z_{12}}\partial_{z_{21}}, \\ d_{34}(k) = (z_{11}z_{22} - z_{12}z_{21})D - (k+1)E + k(k+1), \\ d_{13}(k) = -(k+1)\partial_{z_{12}} - z_{21}D, \\ d_{24}(k) = \quad (k+1)\partial_{z_{21}} + z_{12}D, \\ d_{14}(k) = \quad (k+1)\partial_{z_{11}} - z_{22}D, \\ d_{23}(k) = -(k+1)\partial_{z_{22}} + z_{11}D, \end{cases} \tag{38}$$

where E denotes the scaling operator

$$E = z_{11}\partial_{z_{11}} + z_{12}\partial_{z_{12}} + z_{21}\partial_{z_{21}} + z_{22}\partial_{z_{22}}.$$

Combining (37) and (38), we obtain

$$\begin{aligned}\omega_{k,l} = {} & D\otimes\big((z_{11}z_{22}-z_{12}z_{21})D-(l+1)E+l(l+1)\big)\\ & +\big((z_{11}z_{22}-z_{12}z_{21})D-(k+1)E+k(k+1)\big)\otimes D\\ & +\big((k+1)\partial_{z_{12}}+z_{21}D\big)\otimes\big((l+1)\partial_{z_{21}}+z_{12}D\big)\\ & +\big((k+1)\partial_{z_{21}}+z_{12}D\big)\otimes\big((l+1)\partial_{z_{12}}+z_{21}D\big)\\ & -\big((k+1)\partial_{z_{11}}-z_{22}D\big)\otimes\big((l+1)\partial_{z_{22}}-z_{11}D\big)\\ & -\big((k+1)\partial_{z_{22}}-z_{11}D\big)\otimes\big((l+1)\partial_{z_{11}}-z_{22}D\big).\end{aligned} \tag{39}$$

4.2.3. Transvectant of order $r=1$—generalized Jacobian

For all $f\in\mathcal{P}_k^\Gamma$ and $g\in\mathcal{P}_l^\Gamma$, we obtain

$$[f,g]_1=\mu\circ\omega_{k,l}^{(1)}(f\otimes g), \tag{40}$$

the operator $\omega_{k,l}^{(1)}$ being obtained from $\omega_{k,l}$, by removing the part annihilated by μ. We find

$$\omega_{k,l}^{(1)}=l(l+1)\Delta_{2,0}(D)-(k+1)(l+1)\Delta_{1,1}(D)+k(k+1)\Delta_{0,2}(D). \tag{41}$$

4.2.4. Transvectant of order $r=2$—generalized Hessian

For all $f\in\mathcal{P}_k^\Gamma$ and $g\in\mathcal{P}_l^\Gamma$, we obtain

$$[f,g]_2=\mu\circ\omega_{k,l}^{(2)}(f\otimes g), \tag{42}$$

the operator $\omega_{k,l}^{(2)}$ being obtained from $\omega_{k-1,l-1}\circ\omega_{k,l}$, by removing the part annihilated by μ. We find

$$\begin{aligned}\omega_{k,l}^{(2)} = {} & l^2(l-1)(l+1)\Delta_{4,0}\big(D^2\big)-kl(l-1)(l+1)\Delta_{3,1}\big(D^2\big)\\ & +kl(k+1)(l+1)\Delta_{2,2}\big(D^2\big)-kl(k-1)(k+1)\Delta_{1,3}\big(D^2\big)\\ & +k^2(k-1)(k+1)\Delta_{0,4}\big(D^2\big)-6kl(k+l)\Delta_{2,0}(D)\Delta_{0,2}(D).\end{aligned} \tag{43}$$

5. Towards new transvectant operators

The aim of this section is to derive an infinitesimal criterion to compute new differential operators playing the role of the operator $\omega_{k,l}$ in the preceding section.

5.1. Infinitesimal generators

Consider the morphism of Lie algebras induced by the group morphism ρ_k defined by (6):

$$d\rho_k:\mathfrak{sl}(2n,\mathbb{C})\longrightarrow \mathfrak{end}\,\mathcal{P}. \tag{44}$$

This morphism $d\rho_k$ is none other than the differential of ρ_k at the identity.

Following Sophus Lie, we shall use the language of infinitesimal transforms. Let $J \in \mathfrak{sl}(2n, \mathbb{C})$ and let

$$\gamma = \exp(\varepsilon J) = I + \varepsilon J + O\big(\varepsilon^2\big)$$

denote an element of $\mathrm{SL}(2n, \mathbb{C})$ expanded near the identity. The infinitesimal generator $V_k := d\rho_k(J)$ attached to J can be obtained by computing a first order expansion of $\rho_k(\gamma)$:

$$\rho_k(\gamma) f = f + \varepsilon\, V_k f + O\big(\varepsilon^2\big). \tag{45}$$

Example. Let $\{J^-,\ J^0,\ J^+\}$ a basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. For instance:

$$J^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J^0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \qquad J^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{46}$$

In genus $n = 1$, the computation of

$$\rho_k(\gamma) f(z) = f\left(\frac{az+b}{cz+d}\right)(cz+d)^k \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} := I + \varepsilon J + O\big(\varepsilon^2\big),$$

where $J \in \{J^-, J^0, J^+\}$, yields

$$V_k^- = \partial_z, \qquad V_k^0 = -\frac{k}{2} + z\partial_z, \qquad V_k^+ = kz - z^2\partial_z. \tag{47}$$

More generally the computation of $\rho_k(\gamma) f(Z)$ for an infinitesimal transform

$$\gamma := \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \mathrm{I} + \varepsilon \begin{pmatrix} \dot{A} & \dot{B} \\ \dot{C} & \dot{D} \end{pmatrix} + O\big(\varepsilon^2\big) \tag{48}$$

yields

$$\rho_k(\gamma) f(Z) = f\big((AZ+B)(CZ+D)^{-1}\big) \times \det(CZ+D)^k \tag{49}$$

$$= f(Z + \varepsilon P) \times \big(1 + \varepsilon k \operatorname{tr}(\dot{C}Z + \dot{D})\big) + O\big(\varepsilon^2\big) \tag{50}$$

upon letting $P := \dot{A}Z + \dot{B} - Z(\dot{C}Z + \dot{D})$. We thus obtain the infinitesimal generator:

$$V_k = \sum_{1 \leqslant i,j \leqslant n} P_{ij}\partial_{z_{ij}} + k \operatorname{tr}(\dot{C}Z + \dot{D}). \tag{51}$$

Example. Let us compute the infinitesimal generators attached to the group $\mathrm{Sp}(2n, \mathbb{R})$ for $n = 2$. The conditions (11) become by applying the transform (48)

$$\dot{B} = \dot{B}^t, \qquad \dot{C} = \dot{C}^t, \qquad \dot{D} = -\dot{A}^t. \tag{52}$$

Thus $\mathrm{Sp}(4,\mathbb{R})$ is a 10 parameter Lie group. A matrix $J \in \mathfrak{sp}(4,\mathbb{R})$ is thus of the shape :

$$J = \begin{pmatrix} a_1 & a_2 & a_5 & a_6 \\ a_3 & a_4 & a_6 & a_7 \\ a_8 & a_9 & -a_1 & -a_3 \\ a_9 & a_{10} & -a_2 & -a_4 \end{pmatrix}$$

where the a_i's denote arbitrary real numbers. The 10 matrices $J^1, \ldots, J^{10}$ obtained by letting one the $a_i = 1$ and the others $= 0$, constitute a basis of $\mathfrak{sp}(4,\mathbb{R})$. The formula (51) yields the 10 infinitesimal generators $V_k^i := d\rho_k(J^i)$ for $1 \leqslant i \leqslant 10$:

$$\begin{cases} V_k^1 = 2z_{11}\partial_{z_{11}} + z_{12}\partial_{z_{12}} + z_{21}\partial_{z_{21}} - k, \\ V_k^2 = (z_{21}+z_{12})\partial_{z_{11}} + z_{22}\partial_{z_{12}} + z_{22}\partial_{z_{21}}, \\ V_k^3 = z_{11}\partial_{z_{12}} + z_{11}\partial_{z_{21}} + (z_{21}+z_{12})\partial_{z_{22}}, \\ V_k^4 = z_{12}\partial_{z_{12}} + z_{21}\partial_{z_{21}} + 2\partial_{z_{22}}z_{22} - k, \\ V_k^5 = \partial_{z_{11}}, \\ V_k^6 = \partial_{z_{12}} + \partial_{z_{21}}, \\ V_k^7 = \partial_{z_{22}}, \\ V_k^8 = -z_{11}(z_{11}\partial_{z_{11}} + z_{12}\partial_{z_{12}} + z_{21}\partial_{z_{21}} - k) - z_{12}z_{21}\partial_{z_{22}}, \\ V_k^9 = -(z_{21}+z_{12})z_{11}\partial_{z_{11}} - (z_{11}z_{22}+z_{12}^2)\partial_{z_{12}} \\ \qquad - (z_{11}z_{22}+z_{21}^2)\partial_{z_{21}} - (z_{21}+z_{12})z_{22}\partial_{z_{22}} + k(z_{21}+z_{12}), \\ V_k^{10} = -z_{22}(z_{12}\partial_{z_{12}} + z_{21}\partial_{z_{21}} + z_{22}\partial_{z_{22}} + k) - z_{12}z_{21}\partial_{z_{11}}. \end{cases} \tag{53}$$

5.2. Infinitesimal criterion for invariance

Consider two representations of $\mathrm{GL}(2n,\mathbb{C})$ as per (6). By definition , $(\rho_k \otimes \rho_l)\gamma := \rho_k(\gamma) \otimes \rho_l(\gamma)$.

We have seen that the operator $\Omega : \mathcal{H}_k \otimes \mathcal{H}_l \to \mathcal{H}_{k-1} \otimes \mathcal{H}_{l-1}$ is a $\mathrm{SL}(2n,\mathbb{C})$-invariant differential operator, homogeneous of (bi)degree $(-1,-1)$. This property suffices to show that the transvectant of two invariant functions is also invariant (Prop. 1).

From there we infer that the operator $\omega_{k,l}$ is $\mathrm{SL}(2n,\mathbb{C})$-invariant, that is to say that the following diagram commutes for all $\gamma \in \mathrm{SL}(2n,\mathbb{C})$:

$$\begin{array}{ccc} \mathcal{P}\otimes\mathcal{P} & \xrightarrow{\omega_{k,l}} & \mathcal{P}\otimes\mathcal{P} \\ {\scriptstyle \rho_k(\gamma)\otimes\rho_l(\gamma)}\downarrow & & \downarrow{\scriptstyle \rho_{k-1}(\gamma)\otimes\rho_{l-1}(\gamma)} \\ \mathcal{P}\otimes\mathcal{P} & \xrightarrow{\omega_{k,l}} & \mathcal{P}\otimes\mathcal{P} \end{array}. \tag{54}$$

We proceed to generalize this property for homogeneous operators of arbitrary bi-degree.

Definition 3. Let $(k_0, l_0) \in \mathbb{Z}^2$ denote a bi-index and Γ a Lie subgroup of $\mathrm{GL}(2n, \mathbb{C})$. An operator $\theta_{kl} : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ is said to be Γ-invariant of (bi)degree (k_0, l_0) if and only if the following diagram commutes for all $\gamma \in \Gamma$:

$$\begin{tikzcd} \mathcal{P} \otimes \mathcal{P} \arrow[r, "\theta_{k,l}"] \arrow[d, "V\rho_k(\gamma) \otimes \rho_l(\gamma)"'] & \mathcal{P} \otimes \mathcal{P} \arrow[d, "\rho_{k+k_0}(\gamma) \otimes \rho_{l+l_0}(\gamma)"] \\ \mathcal{P} \otimes \mathcal{P} \arrow[r, "\theta_{k,l}"] & \mathcal{P} \otimes \mathcal{P} \end{tikzcd} \tag{55}$$

We give below an *infinitesimal version* of this commutation relation.

Lemma 2. *Let $J \in \mathfrak{gl}(2n, \mathbb{C})$ and let $k, l \in \mathbb{Z}$. Let us denote by $V_k = d\rho_k(J)$ and $V_l = d\rho_l(J)$ the infinitesimal generators attached to J for the representations ρ_k and ρ_l. Then the infinitesimal generator $V_{k,l}$ attached to J for the representation $\rho_k \otimes \rho_l$ is $V_{k,l} = V_k \otimes 1 + 1 \otimes V_l$.*

Proof.

$$\begin{aligned} \big((\rho_k \otimes \rho_l)(I + \varepsilon J)\big)(f \otimes g) &= (f + \varepsilon V_k f) \otimes (g + \varepsilon V_l g) \\ &= f \otimes g + \varepsilon\, (V_k \otimes 1 + 1 \otimes V_l)(f \otimes g) + O\big(\varepsilon^2\big). \qquad \square \end{aligned}$$

Proposition 3. *Let Γ denote a Lie subgroup of* $\mathrm{GL}(2n, \mathbb{C})$. *An operator $\theta_{k,l} : \mathcal{P} \otimes \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{P}$ is Γ-invariant of (bi)degree (k_0, l_0)—see diagram (55)—if and only if for all elements J of the Lie algebra of the group Γ:*

$$V_{k+k_0, l+l_0} \circ \theta_{k,l} = \theta_{k,l} \circ V_{k,l} \tag{56}$$

where $V_{k,l}$ is the infinitesimal generator attached to J for the representation $\rho_k \otimes \rho_l$.

Proof. The proof is classical and relies on the fact that a connected Lie group is generated by its one parameter subgroups.

We compute the relation (55) in the case of $\gamma = \exp(\varepsilon J) = I + \varepsilon J + O(\varepsilon^2)$. By Lemma 2, we see that: $\rho_k(\gamma) \otimes \rho_l(\gamma) = I + \varepsilon V_{k,l} + O(\varepsilon^2)$. The relation $(\rho_{k+k_0}(\gamma) \otimes \rho_{l+l_0}(\gamma))\theta_{k,l} = \theta_{k,l}(\rho_k(\gamma) \otimes \rho_l(\gamma))$ therefore becomes $(1 + \varepsilon V_{k+k_0,l+l_0})\theta_{k,l} = \theta_{k,l}(1 + \varepsilon V_{k,l}) + O(\varepsilon^2)$. The relation (56) is thus seen to be the infinitesimal version of the relation (55). $\square$

6. Conclusion and open problems

We have shown that the classical theory of transvectant operators acting on functions of two complex variables can be extended to a transvectant acting on functions of two matrix variables. We have stressed the central role played by the operators $\omega_{k,l}$, the invariance of which can characterized by the commutative diagram (54). These operators are obtained so far by complex determinantal manipulations.

When the genus $n > 2$, the transvectants become very complicated, and one should try to better understand their explicit formulae. We are currently experimenting with the relation (56) in the hope of obtaining new invariant operators and thereby new Rankin–Cohen operators for Siegel modular forms. It would also be worth investigating possible connections between our results and the conformally–invariant transvectants recently introduced by Ovsienko and Redou [13].

Finally, we remark that the symplectic transvectants appear in the higher dimensional Moyal bracket, [2,9] which arises in quantum mechanics as the essentially unique quantum deformation of the classical Poisson bracket. Further developments of the connections with quantization, along the lines outlined in [12] in the genus $n = 1$ case, are worth developing.

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